A VARIATIONAL METHOD FOR THE CALCULATION OF TEMPERATURE FIELDS IN STRUCTURAL ELEMENTS

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We examine the application of the variational Kantorovich process for an approximate solution of the two-dimensional nonlinear heat-conduction problem. We provide a comparison with a computer-derived solution.

Let the functional ($M \in v$) be defined in some region v with the boundary S (Fig. 1):

$$I[T(M)] = \int_v \left[(\lambda \nabla T)^2 - 2q_v \int_0^T \lambda dT \right] dv + 2 \int_S \sum_{m=1}^{k} \gamma_m f_m dS. \tag{1}$$

Let us equate the variation of functional (1) to zero:

$$\delta I = 2\int_v \left[\lambda \frac{\partial \lambda}{\partial T} (\nabla T)^2 \delta T + \lambda^2 \nabla T \nabla (\delta T) - q_v \lambda \delta T \right] dv + 2\int_S \sum_m \frac{\partial f_m}{\partial T} \gamma_m \delta T dS = 0. \tag{2}$$

By transformation and with the Ostrogradskii formula we obtain

$$\int_v [\nabla (\lambda \nabla T) + q_v] \delta T \lambda dT - \int_S \left[\lambda \nabla T + \sum_m \frac{\gamma_m}{\lambda} \frac{\partial f_m}{\partial T} \right] \delta T \lambda dS = 0. \tag{3}$$

Since $\lambda > 0$, we come to the steady-state thermal problem with boundary conditions of the general type:

$$\nabla (\lambda \nabla T) + q_v = 0, \tag{4}$$

$$\left[\lambda \frac{\partial T}{\partial n} + \sum_m \frac{\gamma_m}{\lambda} \frac{\partial f_m}{\partial T} \right]_S = 0. \tag{5}$$

Having selected the coordinate system for the function $u_i(M)$, we will seek its approximate solution among the class of functions satisfying the boundary conditions (5) in the form

$$T_n = \sum_{i=1}^{n} s_i u_i + u_s, \tag{6}$$

and here $s_i(x_k)$ is determined from the condition of minimizing the original functional (1).

To construct the coordinate system we use the function $u_0(N)$, which is a solution of the problem ($L \subset S$, $D \subset v$):

$$\nabla (\lambda \nabla u_0) + q_v(N) = 0, \tag{7}$$

$$\left[\lambda \nabla u_0 + \sum_m \frac{\gamma_m}{\lambda} \frac{\partial f_m}{\partial T} \right]_\Gamma = 0, \tag{8}$$

in which λ and f_m are regarded as the function $u_0(N)$.

For the sake of determinacy, let us examine the following problem of finding the temperature field in a cylindrical heat-evolving element with arbitrarily positioned sources within the volume (Fig. 2):

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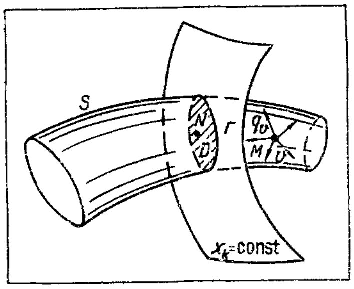

Fig. 1

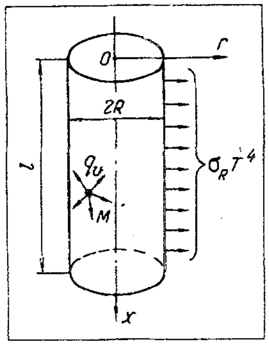

Fig. 2

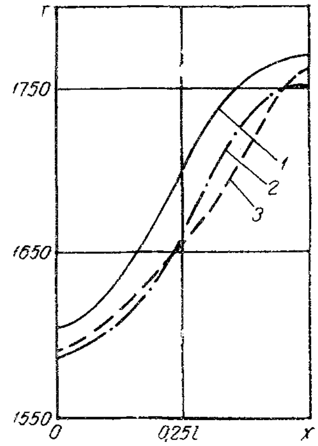

Fig. 3

Fig. 1. Region for which the original functional is determined.

Fig. 2. Theoretical diagram for the cooling of a fuel element in the case of cylindrical symmetry.

Fig. 3. Found axial temperature distribution in fuel elements (T, °K): 1) first approximation; 2) second approximation; 3) electronic digital computer.

$$\nabla [\lambda_0 (1+\beta T) \nabla T] + q_0 w_x (x) w_r (r) = 0, \tag{9}$$

$$[T'_x]_{x=0,\, x=l} = 0, \quad [\lambda_0 (1+\beta T) T'_r + \sigma_R T^4]_{r=R} = 0. \tag{10}$$

It is not difficult to see that the thermal problem (9)-(10) is equivalent to the variational problem for functional (1) when

$$[\gamma_1]_{r=R} = \sigma_R, \quad [f_1]_{r=R} = \lambda_0 \int\limits_0^T T^4 (1+\beta T)\, dT.$$

We turn to the dimensionless variables

$$\rho = \frac{r}{R}, \quad \psi = \frac{x}{R}, \quad t = 1 + 2\beta \int\limits_0^T (1+\beta T)\, dT \tag{11}$$

and the notations

$$\sigma = \frac{2\sigma_R R}{\lambda_0 \beta^3}, \quad q = \frac{2 q_0 R^2 \beta}{\lambda_0}. \tag{12}$$

The original problem is then written in the form

$$\Delta t + q w_\psi w_\rho = 0, \tag{13}$$

$$[t'_\psi]_{\psi=0,\, \psi=l/R} = 0, \quad [t'_\rho + \sigma (\sqrt{t} - 1)^4]_{\rho=1} = 0. \tag{14}$$

According to (6), we seek the approximate solution in the form

$$t_n = u_0 (\rho) \left[s_0 (\psi) + \sum_{i=1}^{n} \alpha_i s_i (\psi) \right], \tag{15}$$

where $\alpha_{2i-1} = \rho^{2i-1}$ and $\alpha_{2i} = (1-\rho)^{2i}$.

In accordance with our original idea, the function $u_0(\rho)$ is defined as the solution of the problem

$$(\rho u'_0)' + \rho w_\rho = 0, \tag{16}$$

$$[u'_0 + \sigma (\sqrt{u_0} - 1)^4]_\rho = 1, \tag{17}$$

which has the form

$$u_0 = -\Phi(\rho) + C, \quad \Phi(\rho) = \int \frac{1}{\rho} \int w_\rho \rho d\rho, \tag{18}$$

where C is a constant determined from (17).

Having determined the total heat release q_{tot} in a cylinder of unit length for fixed ψ and an averaged heat-transfer coefficient Ω (from the heat-balance condition), we find $s_0(\psi)$ as the solution for the problem of a fuel element, i.e.,

$$s_0'' - 2\Omega s_0 + q_\pi w_\psi = 0, \; s_0'(0) = s_0'(l/R) = 0. \tag{19}$$

Since the approximate solution is sought among the class of functions satisfying the boundary conditions, the original functional (1) in the notation of (11) and (12) assumes the form

$$I = \int\limits_{\psi=0}^{l/R} \int\limits_{\rho=0}^{1} \left[(t_\rho')^2 + (t_\psi')^2 - 2qtw_\psi w_\rho\right] \rho d\rho d\psi. \tag{20}$$

We will substitute the expression for t_n from (15) into the appropriate minimization of functional (20), we will integrate with respect to ρ, and we will introduce the dimensionless coefficient of the Euler system of equations

$$a_{ij} = \int\limits_0^1 u_0^2 \alpha_i \alpha_j \rho d\rho, \quad c_i = \int\limits_0^1 w_\rho \alpha_i \rho d\rho, \quad f_i = \int\limits_0^1 u_0^2 \alpha_i \rho d\rho,$$

$$b_{ij} = \int\limits_0^1 \left(u_0'^2 \alpha_i \alpha_j + 2u_0' u_0 \alpha_i \alpha_j + u_0^2 \alpha_i \alpha_j\right) \rho d\rho, \tag{21}$$

$$d_i = \int\limits_0^1 \left(u_0'^2 \alpha_i + u_0' u_0 \alpha_i'\right) \rho d\rho, \quad h_i = \int\limits_0^1 \sigma \left(u_0' \alpha_i + u_0 \alpha_i'\right) \rho^2 d\rho.$$

After integration we turn to the one-dimensional functional which must achieve a minimum for the family of functions determined from the system

$$\sum_i \left(a_{ki} s_i'' - b_{ki} s_i\right) = \varphi_k; \quad i, \; k = 1, \; 2, \; \ldots, \; n-1, \tag{22}$$

where $\varphi_k = d_k s_0 + h_k - qc_k w_\psi - f_k s_0''$ is a known function.

With this selection of s_0 we can assume that

$$(s_{2i-1}/s_0)^m \ll 1 \; (m = 2, \; 3, \; \ldots). \tag{23}$$

Then, substituting (15) into (14), with consideration of (23) we obtain the closing condition which is linear with respect to s_{2i-1}.

As an example we determined the temperature distribution in a fuel element. This was compared with the solution obtained on a digital computer. The results shown in Fig. 3 show that the accuracy is sufficient for engineering calculations (3-4%).

NOTATION

T	is the temperature;
R, l	are the radius and the length of the cylinder;
x, r	are the axial and radial coordinates;
λ	is the coefficient of thermal conductivity;
q	is the specific heat release;
f, γ	are functions characterizing the form of the boundary conditions;
w	is a function characterizing the distribution of the internal sources;
u	is a coordinate function;
s	is the minimizing function;
t	is the dimensionless function of temperature;
ψ, ρ	are dimensionless axial and radial coordinates;
σ	is the reduced radiation factor.

Symbols

0 denotes initial;

x, ψ denote axial;

r, ρ denote radial;

v denotes volume.

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